COGNITIVE APPRENTICESHIP: MAKING THINKING VISIBLE

BY ALLAN COLLINS, JOHN SEELY BROWN, AND ANN HOLUM

IN ANCIENT times, teaching and learning were accomplished through apprenticeship. We taught our children how to speak, grow crops, craft cabinets, or tailor clothes by showing them how and by helping them do it. Apprenticeship was the vehicle for transmitting the knowledge required for expert practice in fields from painting and sculpting to medicine and law. It was the natural way to learn. In modern times, apprenticeship has largely been replaced by formal schooling, except in children's learning of language, in some aspects of graduate education, and in on-the-job training. We propose as alternative model of instruction that is accessible within the framework of the typical American classroom. It is a model of instruction that goes back to apprenticeship but incorporates elements of schooling. We call this model "cognitive apprenticeship" (Collins, Brown, and Newman, 1989).

While there are many differences between schooling and apprenticeship methods, we will focus on one. In apprenticeship, learners can see the processes of work: They watch a parent sow, plant, and harvest crops and help as they are able; they assist a tradesman as he crafts a cabinet; they piece together garments under the supervision of a more experienced tailor. Apprenticeship involves learning a physical, tangible activity. But in schooling, the "practice" of problem solving, reading comprehension, and writing is not at all obvious—it is not necessarily observable to the student. In apprenticeship, the processes of the activity are visible. In schooling the processes of thinking are often invisible to both the students and the teacher. Cognitive apprenticeship is a model of instruction that works to make thinking visible.

In this article, we will present some of the features of traditional apprenticeship and discuss the ways it can be adapted to the teaching and learning of cognitive skills. Then we will present three successful examples—cases in which teachers and researchers have used apprenticeship methods to teach reading, writing, and mathematics.

In the final section, we organize our ideas about the characteristics of successful teaching into a general framework for the design of learning environments, where "environment" includes the content taught, the pedagogical methods employed, the sequencing of learning activities, and the sociology of learning.

TOWARD A SYNTHESIS OF SCHOOLING AND APPRENTICESHIP

Although schools have been relatively successful in organizing and conveying large bodies of conceptual and factual knowledge, standard pedagogical practices render key aspects of expertise invisible to students. Too little attention is paid to the reasoning and strategies that experts employ when they acquire knowledge or put it to work to solve complex or real-life tasks. Where such processes are addressed, the emphasis is on formulaic methods for solving "textbook" problems or on the development of low-level subskills in relative isolation.

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As a result, conceptual and problem-solving knowledge acquired in school remains largely inert for many students. In some cases, knowledge remains bound to surface features of problems as they appear in textbooks and class presentations. For example, Schoenfeld (1985) has found that, in solving mathematics problems, students rely on their knowledge of standard textbook patterns of problem presentation rather than on their knowledge of problem-solving strategies or intrinsic properties of the problems themselves. When they encounter problems that fall outside these patterns, students are often at a loss for what to do. In other cases, students fail to use resources available to them to improve their skills because they lack models of how to tap into those resources. For example, students are unable to make use of potential models of good writing acquired through reading because they have no understanding of how the authors produced such text. Stack with what Scardamalia and Bereiter (1985) call “knowledge-telling strategies,” they are unaware that expert writing involves organizing one’s ideas about a topic, elaborating goals to be achieved in the writing, thinking about what the audience is likely to know or believe about the subject, and so on.

To make real differences in students’ skill, we need both to understand the nature of expert practice and to devise methods that are appropriate to learning that practice. To do this, we must first recognize that cognitive strategies are central to integrating skills and knowledge in order to accomplish meaningful tasks. They are the organizing principles of expertise, particularly in such domains as reading, writing, and mathematics. Further, because expert practice in these domains rests crucially on the integration of cognitive strategies, we believe that it can best be taught through methods that have traditionally been employed in apprenticeship to transmit complex physical processes and skills.

**Traditional Apprenticeship**

In traditional apprenticeship, the expert shows the apprentice how to do a task, watches as the apprentice practices portions of the task, and then turns over more and more responsibility until the apprentice is proficient enough to accomplish the task independently. That is the basic notion of apprenticeship: showing the apprentice how to do a task and helping the apprentice to do it. There are four important aspects of traditional apprenticeship: modeling, scaffolding, fading, and coaching.

In modeling, the apprentice observes the master demonstrating how to do different parts of the task. The master makes the target processes visible, often by explicitly showing the apprentice what to do. But as Lave and Wenger (in press) point out, in traditional apprenticeship, much of the learning occurs as apprentices watch others at work.

Scaffolding is the support the master gives apprentices in carrying out a task. This can range from doing almost the entire task for them to giving occasional hints as to what to do next. Fading is the notion of slowly removing the support, giving the apprentice more and more responsibility.

Coaching is the thread running through the entire apprenticeship experience. The master coaches the apprentice through a wide range of activities: choosing tasks, providing hints and scaffolding, evaluating the activities of apprentices and diagnosing the kinds of problems they are having, challenging them and offering encouragement, giving feedback, structuring the ways to do things, working on particular weaknesses. In short, coaching is the process of overseeing the student’s learning.

The interplay among observation, scaffolding, and increasingly independent practice aids apprentices both in developing self-monitoring and correction skills and in integrating the skills and conceptual knowledge needed to advance toward expertise. Observation plays a surprisingly key role; Lave (1988) hypothesizes that it aids learners in developing a conceptual model of the target task prior to attempting to execute it. Giving students a conceptual model a picture of the whole is an important factor in apprenticeship’s success in teaching complex skills without resorting to lengthy practice of isolated subskills, for three related reasons. First, it provides learners with an advanced organizer for their initial attempts to execute a complex skill, thus allowing them to concentrate more of their attention on execution than would otherwise be possible. Second, a conceptual model provides an interpretive structure for making sense of the feedback, hints, and corrections from the master during interactive coaching sessions. Third, it provides an internalized guide for the period when the apprentice is engaged in relatively independent practice.

Another key observation about apprenticeship concerns the social context in which learning takes place. Apprenticeship derives many cognitively important characteristics from being embedded in a subculture in which most, if not all, members are participants in the target skills. As a result, learners have continual access to models of expertise-in-use against which to refine their understanding of complex skills. Moreover, it is not uncommon for apprentices to have access to several masters and thus to a variety of models of expertise. Such richness and variety help them to understand that there may be multiple ways of carrying out a task and to recognize that no one individual embodies all knowledge or expertise. And finally, learners have the opportunity to observe other learners with varying degrees of skill; among other things, this encourages them to view learning as an incrementally staged process, while providing them with concrete benchmarks for their own progress.

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From Traditional to Cognitive Apprenticeship

There are three important differences between traditional apprenticeship and the kind of cognitive apprenticeship we propose.

As we said, in traditional apprenticeship, the process of carrying out a task to be learned is usually easily observable. In cognitive apprenticeship, one seeks to deliberately bring the thinking to the surface, to make it visible, whether it's in reading, writing, problem solving. The teacher's thinking must be made visible to the students and the student's thinking must be made visible to the teacher. That is the most important difference between traditional apprenticeship and cognitive apprenticeship. Cognitive research, through such methods as protocol analysis, has begun to delineate the cognitive and metacognitive processes that comprise expertise. By bringing these tacit processes into the open, students can observe, enact, and practice them with help from the teacher and from other students.

Second, in traditional apprenticeship, the tasks come up just as they arise in the world. Learning is completely situated in the workplace. When tasks arise in the context of designing and creating tangible products, apprentices naturally understand the reasons for undertaking the process of apprenticeship. They are motivated to work and to learn the subcomponents of the task, because they realize the value of the finished product. They retain what they must do to complete the task, because they have seen the expert's model of the finished product, and so the subcomponents of the task make sense. But in school, teachers are working with a curriculum centered around reading, writing, science, math, history, etc. that is, in large part, divorced from what students and most adults do in their lives. In cognitive apprenticeship, then, the challenge is to situate the abstract tasks of the school curriculum in contexts that make sense to students.

Third, in traditional apprenticeship, the skills to be learned inhere in the task itself: To craft a garment, the apprentice learns some skills unique to tailoring; for example, stitching buttonholes. Cabinetry does not require that the apprentice know anything about buttonholes. In other words, in traditional apprenticeship, it is unlikely that students encounter situations in which the transfer of skills is required. The tasks in schooling, however, demand that students be able to transfer what they learn. In cognitive apprenticeship, the challenge is to present a range of tasks, varying from systematic to diverse, and to encourage students to reflect on and articulate the elements that are common across tasks. As teachers present the targeted skills to students, they can increasingly vary the contexts in which those skills are useful. The goal is to help students generalize the skill, to learn when the skill is or is not applicable, and to transfer the skill independently when faced with novel situations.

In order to translate the model of traditional apprenticeship to cognitive apprenticeship, teachers need to:

- identify the processes of the task and make them visible to students;
- situate abstract tasks in authentic contexts, so that students understand the relevance of the work; and
- vary the diversity of situations and articulate the common aspects so that students can transfer what they learn.

We do not want to argue that cognitive apprenticeship is the only way to learn. Reading a book or listening to a lecture are important ways to learn, particularly in domains where conceptual and factual knowledge are central. Active listeners or readers who test their understanding and pursue the issues that are raised in their minds, learn things that apprenticeship can never teach. To the degree that readers or listeners are passive, however, they will not learn as much as they would by apprenticeship, because apprenticeship forces them to use their knowledge. Moreover, few people learn to be active readers and listeners on their own, and that is where cognitive apprenticeship is critical—observing the processes by which an expert listener or reader thinks and practicing these skills under the guidance of the expert can teach students to learn on their own more skillfully.

Even in domains that rest on elaborate conceptual and factual underpinnings, students must learn the practice or art of solving problems and carrying out tasks. And to achieve expert practice, some version of apprenticeship remains the method of choice.

COGNITIVE APPRENTICESHIP TEACHING READING WRITING AND MATHEMATICS

In this section, we will briefly describe three success models of teaching in the foundational domains of reading, writing, and mathematics and how these models embody the basic methods of cognitive apprenticeship. These three domains are
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foundational not only because they provide the basis for learning and communication in other school subjects but also because they engage cognitive and metacognitive processes that are basic to learning and thinking more generally. Unlike school subjects such as chemistry or history, these domains rest on relatively sparse conceptual and factual underpinnings, turning instead on students' robust and efficient execution of a set of cognitive and metacognitive skills. As such, we believe they are particularly well suited to teaching methods modeled on cognitive apprenticeship.

**Reading**

Palincsar and Brown's (1984) **reciprocal teaching** of reading exemplifies many of the features of cognitive apprenticeship. It has proved remarkably effective in raising students' scores on reading comprehension tests, especially those of poor readers. The basic method centers on modeling and coaching students in four strategic skills: formulating questions based on the text, summarizing the text, making predictions about what will come next, and clarifying difficulties with the text. Reciprocal teaching was originally designed for middle school students who could decode adequately but had serious comprehension problems; it can be adapted to any age group. The method has been used with groups of two to seven students, as well as individual students. It is called reciprocal teaching because the teacher and students take turns playing the role of teacher.

The procedure is as follows: Both the teacher and students read a paragraph silently. Whoever is playing the role of teacher formulates a question based on the paragraph, constructs a summary, and makes a prediction or clarification, if any come to mind. Initially, the teacher models this process and then takes the role of teacher over to the students. When students first undertake the process, the teacher coaches them extensively on how to construct good questions and summaries, offering prompts and critiquing their efforts. In this way, the teacher provides scaffolding for the students, enabling them to take on whatever portion of the task they are able to. As the students become more proficient, the teacher fades, assuming the role of monitor and providing occasional hints or feedback. The transcript below shows the kind of scaffolding and group interaction that occurs with children during reciprocal teaching.

**SAMPLE RECIPROCAL TEACHING DIALOGUE**

(from Palincsar, 1986)

_Text from which students are working:

*Crows have another gift. They are great mimics. They can learn to talk and imitate animal sounds. Some have been known to learn 100 words and even whole phrases. They can imitate the squawk of a chicken, the whine of a dog, or the meow of a cat.*

*Games have a certain fascination to crows. In a game of hide and seek, a crow hides in the hollow of a tree and then sounds a distress caw. The others rush to the spot, look around, then flap away. This may be done over and over, after which the young crow pops out of its hiding place and caws gleefully. Far from being annoyed at this, the flock bursts into loud cawing themselves. They seem to like the trick that has been played on them.*

T: Chantel, you're our teacher, right? Why don't you summarize first? Remember, just tell me the most important parts.

S1: Crows have a hundred words they can learn by imitation. They can imitate chickens, the whine of a dog, and cats.

T: Okay. We can shorten that summary a bit.

S2: You could say they can imitate other animals.

T: Oh! Good one! There's a list there, Chantel, did you notice that? It says they can imitate the squawk of a chicken, the whine of a dog or the meow of a cat; and you could call that "animal sounds." Can you ask us a question?

S1: Ain't no questions in here.

S3: The words (sic) that need to be clarified are (sic) "mimics."

S4: That means imitate, right?

T: Right. How did you figure that out, Shirley?

S4: The paragraph.
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- 4 -
T: Show us how somebody could figure out what "mimic" means.

S5: They are great mimics. They can learn to talk and imitate animal sounds.

T: Yes, so the next sentence tells you what it means. Very good, anything else needs to be clarified?

All: No.

T: What about that question we need to ask? (pause) What is the second paragraph about, Chantel?

S1: The games they play.

S3: They do things like people do.

S4: What kinds of games do crows play?

S3: Hide and seek. Over and over again.

T: You know what, Larry? That was a real good comparison. One excellent question could be, "How are crows like people?"

S4: They play hide and seek.

T: Good. Any other questions there?

S2: How come the crows don't get annoyed?

S5: Because they like it, they have fun. If I had a crow, I'd tell him he was it and see what he'd do.

T: Let's summarize now and have some predictions.

S1: This was about how they play around in games.

T: Good for you. That's it. Predictions anyone?

S2: Maybe more tricks they play.

S4: Other games.

T: Maybe. So far, they have told us several ways that crows are very smart; they can communicate with one another, they can imitate many sounds, and they play games. Maybe we will read about another way in which they are smart. Who will be the next teacher?

Reciprocal teaching is extremely effective. In a pilot study with individual students who were poor readers, the method raised their reading comprehension test scores from 15 percent to 85 percent accuracy after about twenty training sessions. Six months later the students were still at 69 percent accuracy; recovering to 85 percent only after one session. In a subsequent study with groups of two students, the scores increased from about 30 percent to 80 percent accuracy, with very little change eight weeks later. In classroom studies of groups of four to seven students, test scores increased form about 40 percent to 80 percent correct, again with only a slight decline eight weeks later. These are very dramatic effects for any instructional intervention.

Why is reciprocal teaching so effective? In our analysis, which reflects in part the views of Palincsar and Brown, its effectiveness depends upon the co concurrence of a number of factors.

First, the method engages students in asset of activities that help them form a new conceptual model of the task of reading. In traditional schooling, students learn to identify reading with the subskills of recognizing and pronouncing words and with the activities of scanning text and saying it aloud. Under the new conception, students recognize that reading requires constructive activities, such as formulating questions and making summaries and predictions, as well as evaluative ones, such

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T: Good. Any other questions there?

S2: How come the crows don't get annoyed?

S5: Because they like it, they have fun. If I had a crow, I'd tell him he was it and see what he'd do.

T: Let's summarize now and have some predictions.

S1: This was about how they play around in games.

T: Good for you. That's it. Predictions anyone?

S2: Maybe more tricks they play.

S4: Other games.

T: Maybe. So far, they have told us several ways that crows are very smart; they can communicate with one another, they can imitate many sounds, and they play games. Maybe we will read about another way in which they are smart. Who will be the next teacher?

Reciprocal teaching is extremely effective. In a pilot study with individual students who were poor readers, the method raised their reading comprehension test scores from 15 percent to 85 percent accuracy after about twenty training sessions. Six months later the students were still at 60 percent accuracy; recovering to 85 percent only after one session. In a subsequent study with groups of two students, the scores increased from about 30 percent to 80 percent accuracy, with very little change eight weeks later. In classroom studies of groups of four to seven students, test scores increased from about 40 percent to 80 percent correct, again with only a slight decline eight weeks later. These are very dramatic effects for any instructional intervention.

Why is reciprocal teaching so effective? In our analysis, which reflects in part the views of Palincsar and Brown, its effectiveness depends upon the co concurrence of a number of factors.

First, the method engages students in asset of activities that help them form a new conceptual model of the task of reading. In traditional schooling, students learn to identify reading with the subskills of recognizing and pronouncing words and with the activities of scanning text and saying it aloud. Under the new conception, students recognize that reading requires constructive activities, such as formulating questions and making summaries and predictions, as well as evaluative ones, such
as analyzing and clarifying the points of difficulty. As Palincsar points out (1987), working with a text in a discussion format is not the same as teaching isolated comprehension skills—like how to identify the main idea. With reciprocal teaching, the strategies students learn are in the service of a larger purpose: to understand what they are and to develop the critical ability to read to learn.

The second factor that we think is critical for the success of reciprocal teaching is that the teacher models expert strategies in a shared problem context of knowing that they will soon undertake the same task. After they have tried to do it themselves, and perhaps had difficulties, they listen with new knowledge about the task. That is, they can compare their own questions or summaries generated by the group. They can reflect on any differences, trying to understand what led to those differences. We have argued elsewhere that this kind of reflection is critical to learning (Collins and Brown, 1988).

Third, the technique of providing scaffolding is crucial in the success of reciprocal teaching for several reasons. Most importantly, it decomposes the task as necessary for the students to carry it out, thereby helping them to see how, in detail, to go about it. For example, in formulating questions, the teacher might want to see if the student can generate a question on his or her own; if not, she might suggest starting with a “Why” question about the agent in the story. If that fails, she might generate one herself and ask the student to reformulate it in his or her own words. In this way, it gets students started in the new skills, giving them a “feel” for the skills and helping them develop confidence that they can do them. With successful scaffolding techniques, students get as much support as they need to carry out the task, but no more. Hints and modeling are then gradually faded out, with students taking on more and more of the task as they become more skillful. These techniques of scaffolding and fading slowly build students’ confidence that they can master the skills required.

The final aspect of reciprocal teaching that we think is critical is having students assume the dual roles of producer and critic. They not only must produce good questions and summaries, but they also learn to evaluate the summaries or questions of others. By becoming critics as well as producers, students are forced to articulate their knowledge about what makes a good question, prediction, or summary. This knowledge then becomes more readily available for application to their own summaries and questions, thus improving a crucial aspect of their metacognitive skills. Moreover, once articulated, this knowledge can no longer simply reside in tacit form. It becomes more available for performing a variety of tasks; that is, it is freed from its contextual binding and can be used in many different contexts.

Writing

Scardamalia and Bereiter (1985; Scardamalia, Bereiter, and Steinbach, 1984) have developed an approach to the teaching of writing that relies on elements of cognitive apprenticeship. Based on contrasting models of novice and expert writing strategies, the approach provides explicit procedural supports, in the form of prompts, that are aimed at helping students adopt more sophisticated writing strategies. Like other exemplars of cognitive apprenticeship, their approach is designed to give students a grasp of the complex activities involved in expertise by explicit modeling of expert processes, gradually reduced support or scaffolding for students attempting to engage in the processes, and opportunities for reflection on their own and others’ efforts.

According to Bereiter and Scardamalia (1987), children who are novices in writing use a "knowledge telling" strategy. When given a topic to write on, they immediately produce text by writing their first idea, then their next idea, and so on, until they run out of ideas, at which point they stop. This very simple control strategy finesse most of the difficulties in composing. In contrast, experts spend time not only writing but also planning what they are going to write and revising what they have written (Hayes and Flower, 1980). As a result, they engage in a process that Scardamalia and Bereiter call "knowledge transforming," which incorporates the linear generation of text but is organized around a more complex structure of goal setting and problem solving.

To encourage students to adopt a more sophisticated writing strategy, Scardamalia and Bereiter have developed a detailed cognitive analysis of the activities of expert writers. This analysis provides the basis for a set of prompts, procedural facilitations that are designed to reduce students’ information-processing burden by allowing them to select from a limited number of diagnostic statements. For example, planning is broken down into five general processes or goals: (a) generating a new idea, (b) improving an idea, (c) elaborating on an idea, (d) identifying goals, and (e) putting ideas into a cohesive whole. For each process, they have developed a number of specific prompts, designed to aid students in their planning, as shown below. These prompts, which are akin to the suggestions made by the teacher in reciprocal teaching, serve to simplify the complex process of elaborating on one’s plans by suggesting specific lines of thinking for students to follow. A set of prompts has been developed for the revision process as well (Scardamalia and Bereiter, 1983, 1985).
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PLANNING CUES FOR OPINION ESSAYS

(From Scardamalia et al., 1984)

NEW IDEA

An even better idea is ...
An important point I haven't considered yet is ...
A better argument would be ...
A different aspect would be ...
A whole new way to think of this topic is ...
No one will have thought of ...

IMPROVE

I'm not being very clear about what I just said so ...
I could make my main point clearer ...
A criticism I should deal with in my paper is ...
I really think this isn't necessary because ...
I'm getting off the topic so ...
This isn't very convincing because ...
But many readers won't agree that ...
To liven this up I'll ...

ELABORATE

An example of this ...
This is true, but it's not sufficient so ...
My own feelings about this are ...
I'll change this a little by ...
The reason I think so ...
Another reason that's good ...
I could develop this idea by adding ...
Another way to put it would be ...
A good point on the other side of the argument is ...

GOALS

A goal I think I could write to ...
My purpose ...

PUTTING IT TOGETHER

If I want to start off with my strongest idea, I'll ...
I can tie this together by ...
My main point is ...

Scardamalia and Bereiter's teaching method, like reciprocal teaching, proceeds through a combination of modeling, coaching, scaffolding, and fading. First, the teacher models how to use the prompts, which are written on cue cards, in generating ideas about a topic she is going to write on. The example below illustrates the kind of modeling done by a teacher during an early phase of instruction. Then the students each try to plan an essay on a new topic using the cue cards, a process the students call "soloing." While each student practices soloing, the teacher as well as other students evaluate the soloist's performance, by, for example, noticing discrepancies between the soloists stated goals (e.g., to get readers to appreciate the difficulties of modern dance) and their proposed plans (to describe different kinds of dance). Students also become involved in discussing how to resolve problems that the soloist could not solve. As in the reciprocal teaching method, assumption of the role either of critic or producer is incremental, with students taking over more and more of the monitoring and
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problem-solving process from the teacher as their skills improve. Moreover, as the students internalize the processes invoked by the prompts, the cue cards are gradually faded out as well.

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A TEACHER MODELS GETTING STARTED

ASSIGNMENT

(Suggested by students)

Write an essay on the topic "Today's Rock Stars Are More Talented than Musicians of Long Ago."

THINKING-ALOUD EXCERPT

I don't know a thing about modern rock stars. I can't think of the name of even one rock star. How about, David Bowie or Mick Jagger ... But many readers won't agree that they are modern rock stars. I think they're both as old as I am. Let's see, my own feelings about this are ... that I doubt if today's rock stars are more talented than ever. Anyhow, how would I know? I can't argue this ... I need a new idea ... An important point I haven't considered yet is ... ah ... well ... what do we mean by talent? Am I talking about musical talent or ability to entertain—to do acrobatics? Hey, I may have a way into this topic. I could develop this idea by ... 

Note: Underlined phrases represent selection from planning cues similar to those shown in the outline for opinion essays.

Scardamalia and Bereiter have tested the effects of their approach on both the initial planning and the revision of student compositions. In a series of studies (Bereiter and Scardamalia, 1987), procedural facilitations were developed to help elementary school students evaluate, diagnose, and decide on revisions for their compositions. Results showed that each type of support was effective, independent of the other supports. And when all the facilitations were combined, they resulted in superior revisions for nearly every student and a tenfold increase in the frequency of idea-level revisions, without any decrease in stylistic revisions. Another study (Scardamalia, et al., 1984) investigated the use of procedural cues to facilitate planning. Students gave the teacher assignments, often ones thought to be difficult for her. She used cues like those shown above to facilitate planning, modeling the process of using the cues to stimulate her thinking about the assignment. Pre- and post-comparisons of think-aloud protocols showed significantly more reflective activity on the part of experimental-group students. Even when prompts were no longer available to them. Time spent in planning increased tenfold. And when students were given unrestricted time to plan, the texts of experimental-group students were judged significantly superior in thought content.

Clearly, Scardamalia and Bereiter's methods bring about significant changes in the nature and quality of student writing. In addition to the methods already discussed, we believe that there are two key reasons for their success. First, as in the reciprocal teaching approach to reading, their methods help students build a new conception of the writing process. Students initially consider writing to be a linear process of knowledge telling. By explicitly modeling and scaffolding expert processes, they are providing students with a new model of writing that involves planning and revising. Most students found this view of writing entirely new and showed it in their comments ("I don't usually ask myself those questions," "I never thought closely about what I wrote," and "They helped me look over the sentence, which I don't usually do"). Moreover, because students rarely, if ever, see writers at work, they tend to hold naive beliefs about the nature of expert writing, thinking that writing is a smooth and easy process for "good" writers. Live modeling helps to convey that this is not the case. The model demonstrates struggles, false starts, discouragement, and the like.

Second, because writing is a complex task, a key component of expertise are the control strategies by which the writer organizes the numerous lines of thinking involved in producing high-quality text. A clear need of student writers, therefore, is to develop more useful control strategies than evidenced in "knowledge telling." Scardamalia and Bereiter's methods encourage this development in an interesting way: the cue cards act to externalize not only the basic processes involved in planning but also to help students to keep track of the higher-order intentions (such as generating an idea, elaborating or improving on an idea, and so on) that organize these basic processes.

Mathematical Problem Solving*

Our third example is Schonfeld's (1983, 1985) method for teaching mathematical problem solving to college students. Like the other two, this method is based on a new analysis of the knowledge and processes required for expertise, where
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**Mathematical Problem Solving**

Our third example is Schoenfeld's (1983, 1985) method for teaching mathematical problem solving to college students. Like the other two, this method is based on a new analysis of the knowledge and processes required for expertise, where
expertise is understood as the ability to carry out complex problem-solving tasks. And like the other two, this method incorporates the basic elements of a cognitive apprenticeship, using the methods of modeling, coaching, and fading and of encouraging student reflection on their own problem-solving processes. In addition, Schoenfeld’s work introduces some new concerns, leading the way toward articulation of a more general framework for the development and evaluation of ideal learning environments.

One distinction between novices and experts in mathematics is that experts employ heuristic methods, usually acquired tacitly through long experience, to facilitate their problem solving. To teach these methods directly, Schoenfeld formulated a set of heuristic strategies, derived from the problem-solving heuristics of Polya (1945). These heuristic strategies consist of rules of thumb for how to approach a given problem. One such heuristic specifies how to distinguish special cases in solving math problems: for example, for series problems in which there is an integer parameter in the problem statement, one should try the cases $n = 1, 2, 3, 4$, and try to make an induction on those cases; for geometry problems, one should first examine cases with minimal complexity, such as regular polygons and right triangles. Schoenfeld taught a number of these heuristics and how to apply them in different kinds of math problems. In his experiments, Schoenfeld found that learning these strategies significantly increased students’ problem-solving abilities.

But as he studied students’ problem solving farther, he became aware of other critical factors affecting their skill, in particular what he calls control strategies. In Schoenfeld’s analysis, control strategies are concerned with executive decisions, such as generating alternative courses of action, evaluating which will get you closer to a solution, evaluating which you are most likely to be able to carry out, considering what heuristics might apply, evaluating whether you are making progress toward a solution, and so on. Schoenfeld found that it was critical to teach control strategies, as well as heuristics.

As with the reading and writing examples, explicit teaching of these elements of expert practice yields a fundamentally new understanding of the domain for students. To students, learning mathematics had meant learning a set of mathematical operations and methods. Schoenfeld’s method is teaching students that doing mathematics consists not only in applying problem-solving procedures but in reasoning about and managing problems using heuristics and control strategies.

Schoenfeld’s teaching employs the elements of modeling, coaching, scaffolding, and fading in a variety of activities designed to highlight different aspects of the cognitive processes and knowledge structures required for expertise. For example, as a way of introducing new heuristics, he models their selection and use in solving problems for which they are particularly relevant. In this way, he exhibits the thinking processes (heuristics and control strategies) that go on in expert problem solving but focuses student observation on the use and management of specific heuristics. The example in the sidebar provides a protocol from one such modeling.

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**A MATHEMATICIAN THINKS OUT LOUD**

*(from Schoenfeld, 1983)*

**Problem**

Let $P(x)$ and $Q(x)$ be two polynomials with "reversed" coefficients:

\[
P(x) = a_0 x^N + a_1 x^{N-1} + \ldots + a_N x + a_0
\]

\[
Q(x) = b_0 x^M + b_1 x^{M-1} + \ldots + b_M x + b_0
\]

where $a_0 \neq 0 \neq b_0$. What is the relationship between the roots of $P(x)$ and those of $Q(x)$? Prove your answer.

**Expert Model**

What do you do when you face a problem like this? I have no general procedure for finding the roots of a polynomial, much less for comparing the roots of two of them. Probably the best thing to do for the time being is to look at some simple examples and hope I can develop some intuition from them. Instead of looking at a pair of arbitrary polynomials, maybe I should look at a pair of quadratics at least I can solve those. So, what happens if

\[
P(x) = ax^2 + bx + c
\]

and

\[
Q(x) = cx^2 + dx + e
\]

expertise is understood as the ability to carry out complex problem-solving tasks. And like the other two, this method incorporates the basic elements of a cognitive apprenticeship, using the methods of modeling, coaching, and
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**A MATHEMATICIAN THINKS OUT LOUD (from Schoenfeld, 1983)**

**Problem**

Let $P(x)$ and $Q(x)$ be two polynomials with "reversed" coefficients:

\[ P(x) \quad Q(x) \quad \ldots \quad + = a + = a \]

\[ n-2 \quad n \quad a \quad a \]

\[ x^n \quad 2 \]

\[ 0 \quad x^2 \quad x^2a \quad x^n \quad a + 1 \]

\[ n-1 \quad a \quad x \quad a \]

\[ n-1 \quad x \quad l \quad x-n-1 \quad a \quad a \]

\[ x^n-1 \quad 0 \]

\[ a \]
where $a_n \neq 0 \neq a_0$. What is the relationship between the roots of $P(x)$ and those of $Q(x)$? Prove your answer.

Expert Model

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$$P(x) = ax^2 + bx + c \text{ and } Q(x) = cx^2 + bx + a?$$
The roots are
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

respectively.

That's certainly suggestive, because they have the same numerator, but I don't really see anything that I can push or that'll generalize. I'll give this a minute or two, but I may have to try something else...

Well, just for the record, let me look at the linear case. If \( P(x) = ax + b \) and \( Q(x) = bx + a \), the roots are \(-b/a\) and \(-a/b\) respectively.

They're reciprocals, but that's not too interesting in itself. Let me go back to quadratics. I still don't have much of a feel for what's going on. I'll do a couple of easy examples, and look for some sort of a pattern. The clever thing to do may be to pick polynomials I can factor, that way it'll be easy to keep track of the roots. All right, how about something easy like 
\[(x + 2)(x + 3)?\]

Then \( P(x) = x^2 + 5x + 6 \), with roots -2 and -3. So, \( Q(x) = 6x^2 + 5x + 1 = (2x + 1)(3x + 1) \), with roots -1/2 and -1/3.

Those are reciprocals too. Now that's interesting.

How about \( P(x) = (3x + 5)(2x - 7) = 6x^2 - 11x - 35 \). Its roots are -5/3 and 7/2.

\( Q(x) = -35x^2 - 11x + 6 = -(35x^2 + 11x - 6) = -(7x - 2)(5x + 3). \)

All right, the roots are 2/7 and -3/5. They're reciprocals again, and this time it can't be an accident. Better yet, look at the factors: they're reversed! What about 
\[ P(x) = (ax + b)(cx + d) = acx^2 + (bc + ad)x + bd \]  
Then
\[ Q(x) = bdx^2 + (ad + bc)x + ac = (bx + a)(dx + c). \]

Aha! It works again, and I think this will generalize...

At this point there are two ways to go. I hypothesize that the roots of \( P(x) \) are the reciprocals of the roots of \( Q(x) \), in general. (If I'm not yet sure, I should try a factorable cubic or two.) Now, I can try to generalize the argument above, but it's not all that straightforward; not every polynomial can be factored, and keeping track of the coefficients may not be that easy. It may be worth stopping, re- phrasing my conjecture, and trying it from scratch:

Let \( P(x) \) and \( Q(x) \) be two polynomials with "reversed" coefficients. Prove that the roots of \( P(x) \) and \( Q(x) \) are reciprocals.

All right, let's take a look at what the problem asks for. What does it mean for some number, say \( r \), to be a root of \( P(x) \)? It means that \( P(r) = 0 \). Now the conjecture says that the reciprocal of \( r \) is supposed to be a root of \( Q(x) \). That says that \( Q(1/r) = 0 \). Strange. Let me go back to the quadratic case, and see what happens.

Let \( P(x) = ax^2 + bx + c \), and \( Q(x) = cx^2 + bx + a \). If \( r \) is a root of \( P(x) \), then \( P(r) = ar^2 + br + c = 0 \). Now what does \( Q(1/r) \) look like?
\[ Q(1/r) = c (1/r)^2 + b(1/r) + a = \frac{c + br + ar^2}{r^2} = P(r) = 0 \]
when he brings to the surface reasoning processes that are normally covert.

The roots are

\[-b \pm \sqrt{b^2 - 4ac} \over 2a\]

\[-b \pm \sqrt{b^2 - 4ac} \over 2c\]

respectively.

That's certainly suggestive, because they have the same numerator, but I don't really see anything that I can push or that'll generalize. I'll give this a minute or two, but I may have to try something else....

Well, just for the record, let me look at the linear case. If \( P(x) = ax + b \) and \( Q(x) = bx + a \), the roots are \(-b/a\) and \(-a/b\) respectively.

They're reciprocals, but that's not too interesting in itself. Let me go back to quadratics. I still don't have much of a feel for what's going on. I'll do a couple of easy examples, and look for some sort of a pattern. The clever thing to do may be to pick polynomials I can factor; that way it'll be easy to keep track of the roots. All right, how about something easy like \((x + 2)(x + 3)\)?

Then \( P(x) = x^2 + 5x + 6 \), with roots \(-2\) and \(-3\). So, \( Q(x) = 6x^2 + 5x + 1 = (2x + 1)(3x + 1) \), with roots \(-1/2\) and \(-1/3\).

Those are reciprocals too. Now that's interesting.

How about \( P(x) = (3x + 5)(2x - 7) = 6x^2 - 11x - 35 \)? Its roots are \(-5/3\) and \(7/2\);

\( Q(x) = -35x^2 - 11x + 6 = -(35x^2 + 11x - 6) = -(7x - 2)(5x + 3) \).

All right, the roots are \(2/7\) and \(-3/5\). They're reciprocals again, and this time it can't be an accident. Better yet, look at the factors: they're reversed! What about

\[ P(x) \ (ax + b)(cx + d) = acx^2 + (bc + ad)x + bd \ ? \text{ Then} \]

\[ Q(x) \ bdx^2 + (ad + bc)x + ac = (bx + a)(dx + c) \].

Aha! It works again, and I think this will generalize.... At this point there are two ways to go. I hypothesize that the roots of \( P(x) \) are the reciprocals of the roots of \( Q(x) \), in general. (If I'm not yet sure, I should try a factorable cubic or two.) Now, I can try to generalize the argument above, but it's not all that straightforward; not every polynomial can be factored, and keeping track of the coefficients may not be that easy. It may be worth stopping, re-phrasing my conjecture, and trying it from scratch:

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All right, let's take a look at what the problem asks for. What does it mean for some number, say \( r \), to be a root of \( P(x) \)? It means that \( P(r) = 0 \). Now the conjecture says that the reciprocal of \( r \) is supposed to be a root to \( Q(x) \). That says that \( Q(1/r) = 0 \). Strange. Let me go back to the quadratic case, and see what happens.

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\[ Q(1/r) = c \ (1/r)^2 + b(1/r) + a = c + br + ar^2 = P(r) = 0 \]

\[ r^2 \]

\[ -10 - \]